

A Fractiles Perspective to the Joint Price/Quantity Newsvendor Model

Gal Raz

Australian Graduate School of Management, University of New South Wales, Sydney, New South Wales 2052, Australia,
galraz@agsm.edu.au

Evan L. Porteus

Graduate School of Business, Stanford University, Stanford, California 94305, eporteus@leland.stanford.edu

Pricing and quantity decisions are critical to many firms across different industries. We study the joint price/quantity newsvendor model where only a single quantity and price decision is made, such as a fashion or holiday product that cannot be replenished and where the price is advertised nationally and cannot be changed. Demand is uncertain and sensitive to price. We develop a method for easily finding the optimal price and quantity that applies to more general cases than the usual one in which uncertainty is either additive, multiplicative, or a combination of the two. We represent a quantity by its fractile of the probability distribution of demand for a given price. We use a standard approach to approximating a given distribution with a finite number of representative fractiles and assume that these fractile functions are piecewise linear functions of the price. We identify effects that are not usually seen in a joint price/quantity newsvendor model. For example, although the optimal quantity is a decreasing function of the unit cost, the optimal price can be nonmonotone in the unit cost and we shed insight into why. We illustrate that using a simplified structure of demand uncertainty can result in substantially lower profits.

Key words: pricing; simultaneous production planning; newsvendor model; supply chain management

History: Accepted by Candace A. Yano, operations and supply chain management; received November 13, 2003. This paper was with the authors 1 year and 3 months for 3 revisions.

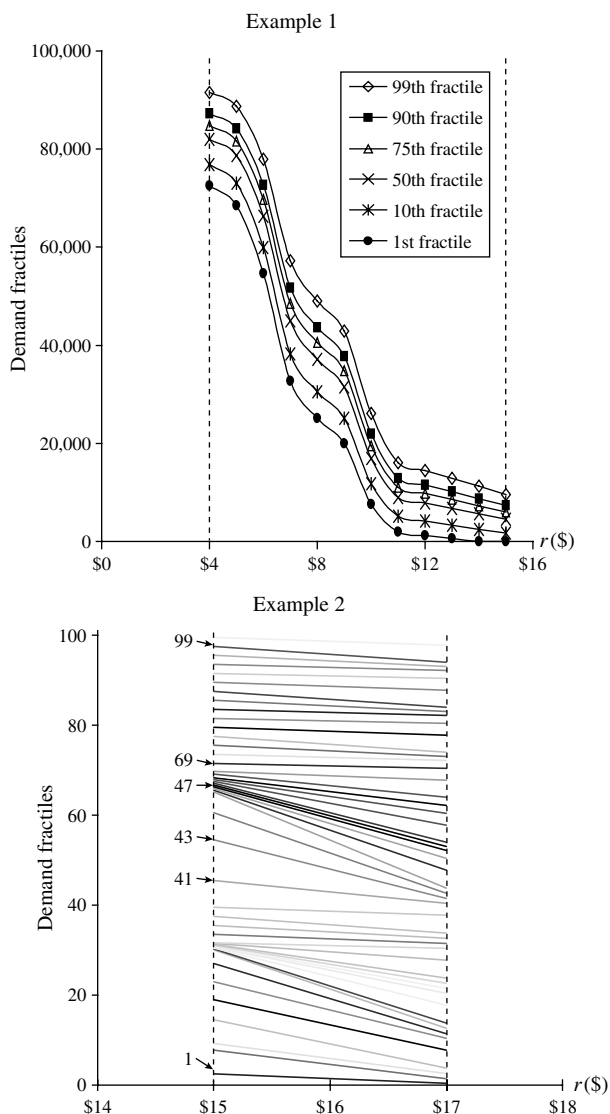
1. Introduction

Harry Potter and the Half-Blood Prince, which was released in July 2005, broke all publishing records. In 24 hours, the book sold 6.9 million copies in the United States alone and a total of 9 million copies worldwide, making it the fastest-selling book in history. This made its U.S. publisher (Scholastic Inc.) rush 2.7 million additional copies into print on top of the 10.8 million copies that were initially printed (*USA Today* 2005). The release of the book also sparked a price war among booksellers and retailers across the United Kingdom (and to a lesser extent in the United States) to try to gain crucial market share (*CBS News* 2005). This is an example of a case where each firm needed to make a simultaneous selection of unit price and stock level in preparation for the dramatic first few days of a selling season, and there was a mismatch between quantity and price (either too little was ordered or the prices were set too low). We address a simpler environment, the joint price/quantity newsvendor model, where no replenishment or price adjustment can take place after a mismatch has been observed, and thus the consequences of a mismatch may be more significant

because there are fewer options available to ameliorate its consequences.

Most of the research on the joint price/quantity newsvendor problem assumes that demand variability either stays constant or decreases as the price is increased. However, there are situations in which these assumptions are not met and a solution is needed. For example, when a company launches a new product, such as software giant Microsoft launching its Xbox 360 at the end of 2005 (*Business Week* 2005), it must estimate the relationship between the price it sets and the demand distribution that it will face. One plausible way to do this is to use conjoint (trade-off) analysis (Green and Srinivasan 1990). We use an example throughout the paper that is based on this approach, where the lowest demand variability comes at either very high or very low prices. The opposite could be the case in situations where there is a good understanding of the market for a middle range of prices and more uncertainty for prices that are outside that range, on either the high or low side. That middle range could represent either the prices of other competing products in the marketplace or a traditional price point that the firm has used in the past. The additional uncertainty can arise on the part of consumers as well as the firm.

Figure 1 Demand Fractiles as Functions of the Price, r



We provide a tractable method for finding the solution, considering price to be a continuous variable, representing the demand distribution in terms of a finite number of its fractiles, and approximating them as piecewise linear functions of the price. Figure 1 illustrates our approach with two examples, which we use throughout the paper to illustrate our results.

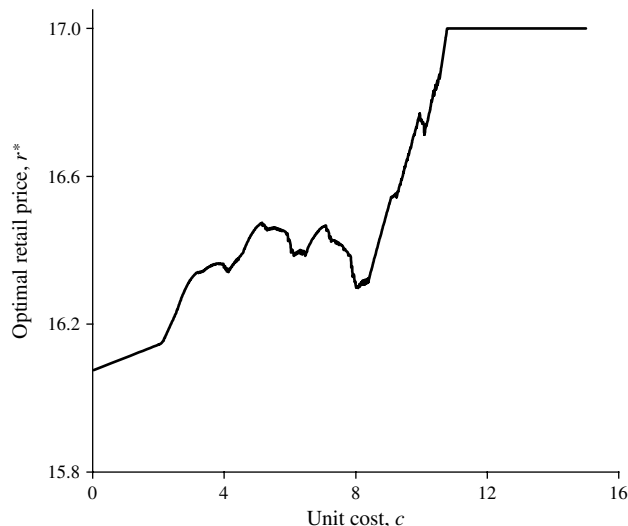
In the first example, which is based on empirical data gathered using conjoint analysis (discussed in more detail in §5), the fractiles are nonlinear in the price on the interval [4, 15]. In particular, the demand variability first increases in price and then decreases in it, which occurs in this example because when either very few or almost all of the target market is expected to buy, there is little variability (the fractiles are close to each other), while there is the most variability when half of the target market is expected to buy. In the second example, the fractiles are lin-

ear over the interval [15, 17]. However, as the price increases, some fractiles get closer together while others diverge from each other. Such a phenomenon may arise when there are different market segments with heterogeneous uncertain responses to changes in price. Only a few of the fractiles are shown in the first example, while 50 of them (1%, 3%, ..., 99%) are shown in the second. Our approach is to approximate a general response surface by representing the demand distribution by a finite number, N , of representative fractiles, each of them a piecewise linear function of the price.

The literature provides conditions under which the model is well behaved in various ways. For example, Ha (2001) shows that in the additive case with a PF_2 distribution, the optimal price increases in the unit cost. However, Zabel (1970) warns that the model is not necessarily well behaved in general: the optimal price may not be a monotone function of the stock level or the unit holding cost even in the multiplicative uncertainty case. Figure 2 reasserts that warning by showing that the optimal retail price is a non-monotone function of the unit cost in a quite accurate approximate solution of Example 2. (Sections 4–6 discuss this problem further, which is specified in detail in the appendices to this paper.)

Such behavior signals tractability problems in the analysis of the continuous model. Our approach, although not elegant, assures tractability: using the standard result for the newsvendor model, we show that for each price, the quantity to be stocked (purchased or produced) falls on one of the N fractile functions. Section 5 shows that only a subset of those fractiles are eligible for optimality. Thus, we first find the optimal solutions for each fractile problem that is eligible for optimality and, second, pick the one that

Figure 2 Optimal Price as a Function of the Cost



achieves the highest expected profit by enumeration. In §5, we also illustrate that the objective function may have multiple local optima, and thus, we should not be surprised that an enumerative method is used.

The linear fractile problem, corresponding to restricting consideration to a single piece of the general piecewise linear setting and a given fractile, is a critical building block in our analysis and is very well behaved: its objective function is concave and its optimal solution (price, quantity purchased, and resulting return) is given explicitly. This result facilitates understanding the drivers of the solution, which can be divided into two main components: the customer component and the cost component. In §4, we elaborate more on these components and their effect on the optimal solution.

Even though, as already indicated, the optimal retail price overall may not be a monotone function of the unit cost, we do show that the optimal quantity overall decreases in the unit cost.

Section 2 discusses the related literature. Section 3 formulates the problem. Section 4 analyzes the linear problem by means of the linear fractile problem, and examines the special cases of additive and multiplicative demand that appear frequently in the literature. Section 5 addresses solving the general problem and §6 examines the effect of the problem parameters on the optimal solution. Section 7 illustrates that significant improvements in profits can result from using the correct, more detailed model compared to the additive and multiplicative models. Section 8 contains concluding remarks and possible options for further research.

2. Relationship to the Literature

Marketing and production joint decision making are significant research domains. For a review of this area of research, see Eliashberg and Steinberg (1993). Price is the most common marketing decision taken into account in this type of research. Yano and Gilbert (2003) provide an extensive review of coordinated pricing and production decisions. Models that simultaneously choose stock levels and prices are divided along two dimensions: deterministic versus stochastic and static versus dynamic. Our model is a stochastic static decision model, and, hence, we focus on results for this model in the literature.

Young (1978) presented the following useful form of the demand:

$$D(r) = g(r) + h(r)U, \quad (1)$$

where $g(r)$ and $h(r)$ are deterministic functions that are weakly decreasing in r , the unit price to be selected, and U is a random variable with zero mean. It is straightforward to show, in this relatively general case, that the fractiles are either parallel or get

closer together as r increases. Whitin (1955) was the first to include pricing decisions in a newsvendor model. He assumed additive demand: $D(r) = g(r) + U$ [$h(r) = 1$]. He also assumed that $g(r)$ is linear and U is uniformly distributed. Using a sequential procedure, he determined the optimal stocking level as a function of price and then the corresponding optimal price. Mills (1959) generalized Whitin's formulation to allow $g(r)$ and U to be general. He compared standard (deterministic) monopoly theory to the case where uncertainty is present, and showed that when the marginal (production) cost is constant, the optimal price under uncertainty is lower than its deterministic equivalent. Karlin and Carr (1962) showed that in the multiplicative demand case, where $D(r) = g(r)(U + 1)$ [$g(r) = h(r)$], the optimal price is always greater than its deterministic equivalent. These two results signaled that the behavior of the optimal solution depends on the nature of the uncertainty. Zabel (1970) showed that under mild conditions on the multiplicative demand model, the optimal stock level (after ordering) increases in the initial stock level and decreases in the unit holding cost, and the optimal price decreases in the initial stock level. However, as already indicated, he warned that the optimal price may be nonmonotone in the stock level after ordering and in the unit holding cost.

Petruzzi and Dada (1999) provide a comprehensive review that synthesizes the earlier results for the single-period problem and develop additional results for it. They study the difference between the optimal price and the deterministic optimal price. They explain that the retailer can use the retail price to reduce demand uncertainty and thereby the risk of overstocking and understocking. We discuss these results in relation to our framework in §4.

Although our model is static, it is worth mentioning some of the key contributions to the literature on dynamic models: Karlin and Carr (1962), Zabel (1972), Thowsen (1975), Young (1978), Amihud and Mendelson (1983), Li (1988), Federgruen and Heching (1999), Chan et al. (2005), and Netessine (2006).

3. The Basic Demand Model

A firm must purchase the stock level, y , of a single product at a unit cost c (>0), at the beginning of a single period prior to observing demand for the product. The firm must also set the retail price, r , of the product at that time. The firm faces stochastic demand, $D(r)$, for the product. We approximate the demand distribution using what we call the *fractile approximation approach*, which conducts two types of approximations: (1) discretizing the demand distribution using representative fractiles, and (2) approximating the (possibly nonlinear) fractiles, which are functions of r , by piecewise linear functions, using the

same breakpoints for all fractiles used. In particular, we discretize the demand distribution for each given price r using N discrete outcomes, each arising with a given probability that does not depend on r .

Consider the second example in Figure 1, which shows 50 fractiles of the distribution. These fractiles are representative fractiles (using our fractile approximation approach) in the following approximation: For each price r in the interval, the demand distribution is approximated by a discrete distribution with 50 equally-likely outcomes, namely, those given by each of the 50 given fractile functions. For example, the 1% fractile function gives the smallest possible outcome, the 3% fractile gives the next largest, and so on. Each outcome arises with probability 0.02. This representation is a standard way¹ to approximate a continuous distribution by a discrete one with 50 equally-likely outcomes. If we increase the number of fractiles chosen to represent the continuous distribution, we would get a better approximation, eventually approaching the continuous distribution, under suitable regularity conditions.

As we saw in Example 1, the demand fractiles can be nonlinear in the price, r . It is natural to approximate these with a piecewise linear function, especially because the fractiles can be assumed to be continuous decreasing functions. The next subsection describes the general demand model.

3.1. The General Demand Model

Demand for the product is stochastic and price sensitive: for each price r , there are N possible fractile functions; $D_i(r)$ arises with probability p_i . Unmet demands are lost with a stockout penalty of p , while unsold stock has a unit salvage value of v net of any holding cost, where $v < c$.

We assume that the retail price, r , must be within a prescribed interval $[a_0, a_m]$, where $c < a_0$, and we partition that interval into m pieces, defined by a_j for $j = 0, 1, \dots, m$, where $a_0 < a_1 < \dots < a_m$. In particular, we assume that if $a_{j-1} \leq r \leq a_j$, then $D_i(r) = D_{ij}(r)$, and for each i and j ,

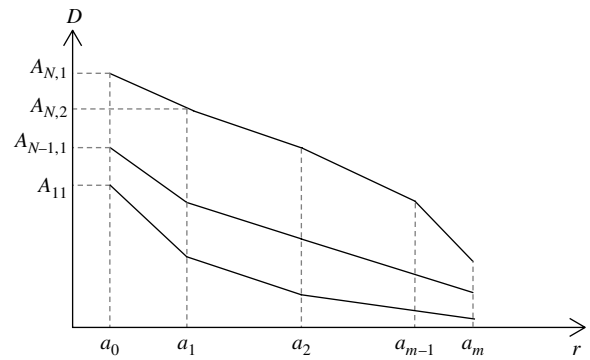
$$D_{ij}(r) = A_{ij} - k_{ij}(r - a_{j-1}), \quad (2)$$

where $A_{ij} > 0$ and $k_{ij} > 0$ for each i and j . That is, fractile function i is linear with slope $-k_{ij}$ on piece j . Because the fractile functions are continuous, then for $j \leq m - 1$,

$$A_{i,j+1} = A_{ij} - k_{ij}(a_j - a_{j-1}) = A_{i1} - \sum_{s=1}^j k_{is}(a_s - a_{s-1}). \quad (3)$$

¹ This is similar to the use of “the fractile method” (Selvidge 1980) and “the percentile approach to fitting distributions and data” (Karian and Dudewicz 2000), where fractiles are used to approximate a continuous distribution when data is available, or “binned kernel density estimation” (Silverman 1986), used in statistics.

Figure 3 The General Demand Model



Because fractiles do not cross one another, we have

$$A_{Nj} \geq A_{N-1,j} \geq \dots \geq A_{1j} \quad \text{for each } j. \quad (4)$$

It is straightforward to show that because we assume that $k_{ij} > 0$, demand is first-order stochastically decreasing in price. The model is illustrated in Figure 3.

3.2. The Additive and Multiplicative Models

Recall from §2 that the additive and multiplicative models correspond to having $h(r) = 1$ and $h(r) = g(r)$, respectively, in $D(r) = g(r) + h(r)U$. Our approach to representing these problems is to discretize the random variable U and approximate $g(r)$ by a piecewise linear function. Figure 4 provides simple illustrations of each model.

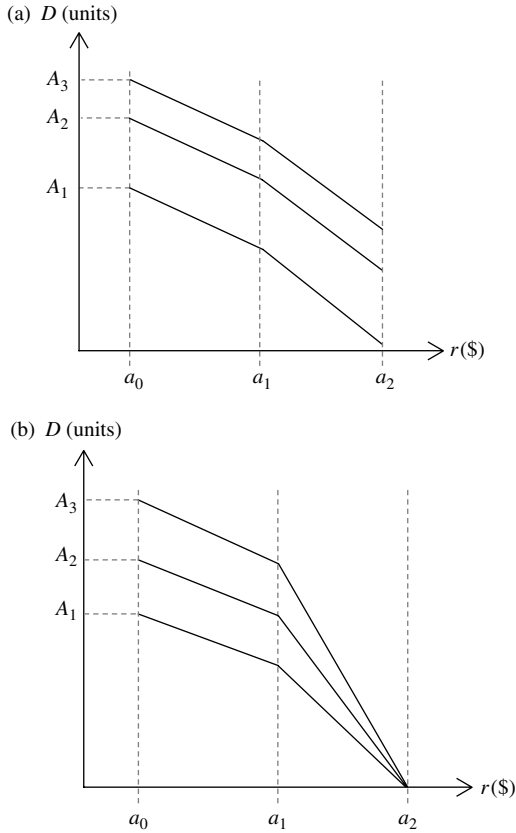
It is easy to show that in the additive case, when the variance of demand is equal for all prices and the coefficient of variation of demand is increasing in the price, the resulting fractiles are parallel: $k_{ij} = k_j$ for each fractile i and piece j . In the multiplicative case, when the variance of demand is decreasing in the price and the coefficient of variation of demand is equal for all prices, the fractile functions get closer as r increases and the slope parameters, k_{ij} , are strictly increasing in i for each j . In both cases, the slope parameters are weakly increasing in i , which implies that the fractiles get (weakly) closer together as r increases, and, hence, the variance of demand is weakly decreasing in the price, r .

4. Introduction to the Linear Case

In this section, we focus on the important case in which all the fractile functions are linear (technically, affine) over a given piece. The optimal solution of the general problem is based on solving a sequence of these. Insights can be drawn by comparing the solution of the linear case to that of a deterministic case. Additional insights can be found by examining the additive and the multiplicative models.

The retail price r is restricted to be in the interval $[a, b]$, where $c < a < b$. Fractile function i takes the

Figure 4 (a) The Additive Demand Model; (b) The Multiplicative Demand Model



form of $D_i(r) = A_i - k_i(r - a)$, where $A_i > 0$ and $k_i > 0$ for each i . We also define R_i as the (extrapolated) intercept of function i with the r axis:

$$R_i := \frac{A_i}{k_i} + a. \tag{5}$$

We interpret R_i as the (extrapolated) *maximum reservation price* at the i th fractile of the distribution. Using definition (3), condition (4) in this case simplifies to

$$A_N \geq A_{N-1} \geq \dots \geq A_1 \tag{6}$$

and

$$\begin{aligned} &A_N - k_N(b - a) \\ &\geq A_{N-1} - k_{N-1}(b - a) \geq \dots \geq A_1 - k_1(b - a). \end{aligned} \tag{7}$$

The expected profits of the firm for stock level y and retail unit price r are

$$\begin{aligned} \Pi(y, r) = &-cy + r \sum_{j=1}^N p_j \min(y, D_j(r)) \\ &+ v \sum_{j=1}^N p_j (y - D_j(r))^+ - p \sum_{j=1}^N p_j (D_j(r) - y)^+, \end{aligned}$$

which simplifies to

$$\begin{aligned} \Pi(y, r) = &(r + p - c)y - (r + p - v) \\ &\cdot \sum_{j=1}^N p_j (y - D_j(r))^+ - p \sum_{j=1}^N p_j D_j(r). \end{aligned} \tag{8}$$

4.1. The Deterministic Version

Consider the *deterministic version* of the problem, in which the firm is facing the expected demand with certainty. In this case, the firm stocks the expected demand given its choice of price r , and the objective function for $r \geq a$ is equal to

$$(r - c)D(r) = (r - c)(\bar{A} - \bar{k}(r - a)), \tag{9}$$

where

$$\bar{A} = \sum_{i=1}^N p_i A_i \tag{10}$$

and

$$\bar{k} = \sum_{i=1}^N p_i k_i. \tag{11}$$

It is straightforward to show that the *deterministic unconstrained optimal price* (relaxing the constraint $a \leq r \leq b$) is given by

$$r_D := \frac{\bar{R} + c}{2}, \tag{12}$$

where \bar{R} is defined analogously to (5) as

$$\bar{R} := \frac{\bar{A}}{\bar{k}} + a. \tag{13}$$

Thus, r_D is the average of the unit cost and the maximum reservation price. Because the objective function in (9) is concave in r , the *deterministic optimal price*, r_D^* , will equal the closest point in $[a, b]$ to r_D .

4.2. Solution of the Linear Fractile Problem

We define the *ith fractile problem* as the problem in which the stock level must fall on the *ith* fractile function, $y = D_i(r)$. Thus, let

$$\Pi_i(r) := \Pi(D_i(r), r) \quad \text{for } i \in \{1, 2, \dots, N\},$$

the expected return from price r for the *ith* fractile problem.

A *linear fractile problem* is a linear problem in which the fractile is specified. That is, it is the *ith* fractile problem for some i , over an interval in which the problem is linear. It forms the basis for solving both the linear case and the general case. It is straightforward to show, using the standard newsvendor result, that for a given price, the optimal quantity will be on one of the fractiles. Thus, one way of solving the problem for the linear case is to first

find the best price and quantity for each fractile problem and then enumerate over the N fractile problems to find the optimal fractile. In §5, we will accelerate this process, but the solution to the linear fractile problem will still play a critical role. For ease of exposition, all proofs are located in the online appendix available on the *Management Science* website (<http://mansci.pubs.informs.org/ecompanion.html>).

THEOREM 1. (a) *The profit function for the i th linear fractile problem, $\Pi_i(r)$, is concave and can be written as*

$$\Pi_i(r) = (r - c_i)\tilde{D}_i(r) - M_i, \quad (14)$$

where

$$\tilde{D}_i(r) = \tilde{A}_i - \tilde{k}_i(r - a), \quad (15)$$

$$M_i = (c - v)[A_i - k_i(\tilde{R}_i - a)] + p[\bar{A} - \bar{k}(\tilde{R}_i - a)], \quad (16)$$

$$c_i := v - p + \frac{k_i(c - v)}{\tilde{k}_i} + \frac{p\bar{k}}{\tilde{k}_i}, \quad (17)$$

$$\tilde{A}_i := A_i \left(1 - \sum_{j=1}^{i-1} p_j\right) + \sum_{j=1}^{i-1} p_j A_j, \quad (18)$$

$$\tilde{k}_i := k_i \left(1 - \sum_{j=1}^{i-1} p_j\right) + \sum_{j=1}^{i-1} p_j k_j, \quad (19)$$

$$\tilde{R}_i := \frac{\tilde{A}_i}{\tilde{k}_i} + a, \quad (20)$$

and $[a, b]$ is the interval over which the model is linear.

(b) *The maximizer, r_i^* of $\Pi_i(\cdot)$ over the segment $[\alpha, \beta]$, where $a \leq \alpha \leq \beta \leq b$, is uniquely given by*

$$r_i^* = \begin{cases} \alpha & \text{if } r_i < \alpha, \\ r_i & \text{if } \alpha \leq r_i \leq \beta, \\ \beta & \text{otherwise,} \end{cases} \quad (21)$$

where

$$r_i := \frac{(\tilde{R}_i + c_i)}{2}. \quad (22)$$

If $r_i^* = r_i$, then the optimal stock level y_i^* is given by

$$y_i^* = \frac{A_i}{2} - \frac{k_i(c_i - a)}{2} + \frac{A_i \sum_{j=1}^{i-1} p_j k_j - k_i \sum_{j=1}^{i-1} p_j A_j}{2\tilde{k}_i}. \quad (23)$$

The reason for distinguishing between the interval $[a, b]$ over which the model is linear and the segment $[\alpha, \beta]$ within that interval over which the price optimization takes place will become clear in §5. We call $\tilde{D}_i(r)$ the *equivalent deterministic demand* for this problem, with height \tilde{A}_i and slope \tilde{k}_i . By (18), the height is a convex combination of the heights of fractile i and the lower fractiles. Similarly, by (19), the slope is the same convex combination of the slopes for fractile i

and the lower fractiles. By (14), this equivalent deterministic problem incorporates a fixed cost, M_i , which we call the *adjusted fixed cost* (and is independent of the price and quantity). We also call r_i the *unconstrained optimal price* and r_i^* the *(constrained) optimal price* for this problem.

We call \tilde{R}_i the *customer component* and c_i the *cost component*. By (22), the unconstrained optimal price is the average of the customer component and the cost component as in the deterministic case. By (18)–(20), the customer component depends solely on the customer environment. The cost component depends on both the cost structure and the customer environment. By (17), the slopes of various fractiles (which are determined by the customer environment) enter into the determination of c_i . However, the latter dependence is often small. For instance, as we shall see shortly, in the case of additive demand, the cost component is independent of the customer environment. Thus, it is convenient to interpret the cost component as being primarily dependent on the cost structure. We therefore can say that the cost structure and customer environment contribute about equal amounts to the determination of the (unconstrained, optimal) price for each given fractile.

Furthermore, by (17)–(22), the (constrained) optimal price, r_i^* , depends on the higher demand outcomes ($i + 1, i + 2$, etc.) only through c_i and that only through \bar{k} . Thus, if all the slope parameters are equal as in the additive case, or if the penalty cost is 0, then the optimal price for the i th fractile problem will not depend on the higher demand outcomes.

4.3. Insights into the (Linear) Additive and Multiplicative Models

Petruzzi and Dada (1999) provide important intuition about the role of variability in explaining optimal pricing. In the additive case, Mills (1959) shows that the optimal price is lower than in the deterministic case. However, in the multiplicative case, Karlin and Carr (1962) show that the optimal price is higher than in the deterministic case. Petruzzi and Dada (1999) suggest that these results may be driven by the incentive to lower variability: in the additive case, a lower price decreases the coefficient of variation and keeps the variance constant, while in the multiplicative case, a higher price decreases the variance, and keeps the coefficient of variation constant. This subsection examines what can be learned about these models when applying our framework to the linear case.

As discussed, the unconstrained optimal price for the i th fractile problem is the average of the customer component and the cost component. The next result shows that in the (linear) additive and multiplicative models, one of those parameters becomes equal to its

deterministic counterpart. Consequently, the relationship between the optimal price and the deterministic price depends solely on the second parameter, and further insight is shed on the results of Mills (1959) and Karlin and Carr (1962).

THEOREM 2. (a) In the (linear) additive model, $c_i = c$, $\tilde{R}_i \leq \bar{R}$, and, hence, $r_i^* \leq r_D^*$ for each i .

(b) In the (linear) multiplicative model, $c_i \geq c$, $\tilde{R}_i = \bar{R}$, and, hence, $r_i^* \geq r_D^*$ for each i .

Theorem 2 shows that in the additive case, the cost component is equal to its deterministic counterpart and independent of the other cost parameters, and, thus, the customer environment determines any differences among the optimal prices for the constrained fractile problems. The cost structure as a whole determines which fractile is optimal, but because each price for each given fractile is less than the deterministic price, the (overall) optimal price must also be less. On the other hand, when demand is multiplicative, the customer component is constant, and thus it is the cost structure that determines any differences among the optimal prices for the constrained fractile problems. Because each such price exceeds the deterministic price, the (overall) optimal price must also. Hence, variability in the problem affects only the customer component in the additive case and only the cost component in the multiplicative case. However, in general, both components may be affected and the nature of the influence may not be predictable, and, thus, it is unclear how the optimal price will compare to the deterministic price.

5. Solution to the General Case

In this section, we discuss our solution procedure for solving the general case and illustrate it using examples. The next theorem helps by showing that regardless of the shapes of the fractile functions, we can

partition the set of feasible prices into at most N intervals, such that when the price is in the i th interval, the optimal amount to stock is on the i th fractile function.

THEOREM 3. If r is a feasible price, then $D_i(r)$ is an optimal stock level if $\gamma_{i-1} \leq r \leq \gamma_i$, where $\gamma_0 := 0$, $\gamma_N := \infty$, and, for $i = 1, 2, \dots, N - 1$,

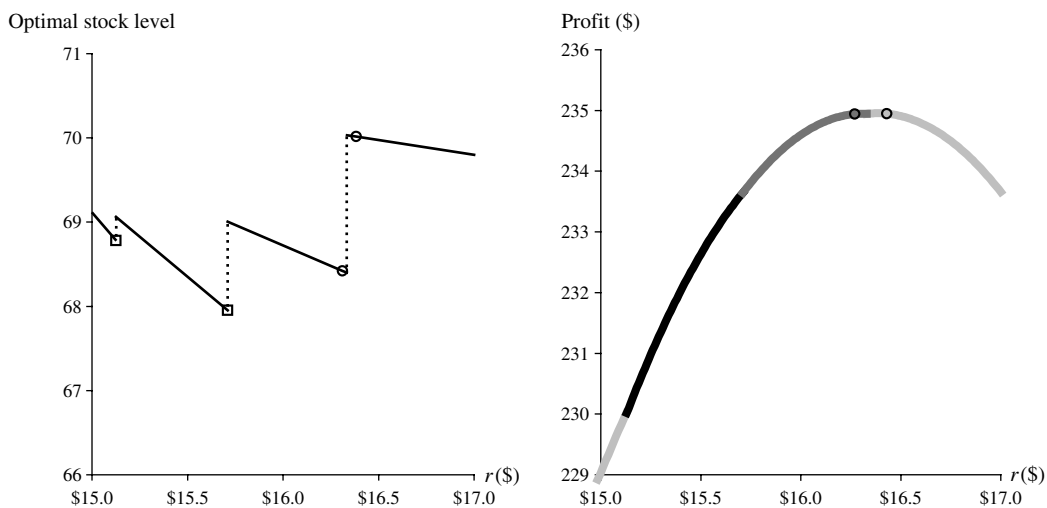
$$\gamma_i := \frac{c - v}{\sum_{j=i+1}^N p_j} - (p - v). \tag{24}$$

The proof follows directly from the standard newsvendor result (e.g., Porteus 2002). The optimal stock level always falls on one of the fractile functions, and when the price increases from one price interval to the next, the optimal amount to stock jumps to the next higher fractile function. Note that at the boundary point γ_i , the optimal profit on the i th fractile function is equal to that of the $i + 1$ st fractile function.

Figure 5 illustrates the application of Theorem 3 to Example 2 using 100 fractile functions for the case of $c = 7.8$, $v = 2$, and $p = 5$. The left panel gives, for each price, the optimal stock level for the optimal fractile identified by Theorem 3. It shows that even though there are 100 different possible fractiles, the optimal stock level can be on only four of them (the 68th, 69th, 70th, and 71st fractiles). The right panel plots the resulting objective function.

The left panel also shows the optimal solution (price and stock level) within each interval where a given fractile is optimal. In (each of) the first two intervals, the optimal solution is at the right end and is shown by a square. In the third and fourth intervals, the optimal solution is in the interior and is shown by a circle, with the corresponding price and objective function value also shown in the right panel. (The optimal solution is the one in the fourth interval.) The following result confirms that it is possible to accelerate the process of identifying the optimal price within

Figure 5 The Optimal Stock Level and Profit as a Function of the Price, r



a segment where the model is linear. (Recall that r_i is the unconstrained optimal price for fractile i , given by (22).)

THEOREM 4. Suppose that $D_i(r)$ is linear on $[a, b]$ for each i .

(a) If $r_i \geq \gamma_i$ and $b > \gamma_i$, then fractile $i + 1$ dominates fractile i .

(b) If $r_i \leq \gamma_{i-1}$ and $a < \gamma_{i-1}$, then fractile $i - 1$ dominates fractile i .

The idea is simple: for example, if $r_i \geq \gamma_i$, then the best price for fractile i within the eligible interval $[\gamma_{i-1}, \gamma_i]$ of optimal prices is γ_i , which, because $b > \gamma_i$, is a feasible price. But the return from fractile $i + 1$ at price γ_i is equal to that from fractile i at that price, and the best return from fractile $i + 1$ over its eligible interval is at least as good as that. Thus, fractile i is dominated.

The left panel of Figure 5 also illustrates Theorem 4: the unconstrained optimal prices for the first and second eligible fractiles are either equal to or larger than the upper limit of their respective eligible interval, and are therefore dominated by the next higher fractile. That is, the third eligible fractile dominates the first two. Thus, we need only compare the objective function values for the third and fourth points to determine the optimal solution.

We now discuss the right panel of Figure 5 in more detail. Theorems 1 and 3 assure that the objective function is concave within each interval where a particular fractile is optimal. The overall objective function is therefore piecewise concave, but not necessarily concave. Example 3 in Figure 6 illustrates this more clearly. Example 3 has three representative fractiles: $D_1(r) = 40 - 4(r - 30)$, $D_2(r) = 65 - 3(r - 30)$, $D_3(r) = 105 - (r - 30)$, $c = 20$, $v = 4$, $p = 1$, $a = 30$, $b = 40$, $p_1 = 0.2$, $p_2 = 0.3$, and $p_3 = 0.5$. Using Theorem 3,

we get that $\gamma_1 = 23$ and $\gamma_2 = 35$. Thus, the 1st fractile cannot be optimal within $[30, 40]$, the 2nd is optimal within $[30, 35]$, and the 3rd is optimal within $[35, 40]$. Figure 6 shows the objective function as a function of the price, r , within $[30, 40]$. There are two local optima, one at $r = 33.72$ and the other at $r = 38.82$, which is also the global optimal solution. In short, the objective function is not concave or even quasi-concave: it is piecewise concave.

5.1. The Solution Procedure

We now present our accelerated, five-step approach to solving the general problem. Recall that we start with m different linear models, one for each piece j ($= 1, 2, \dots, m$) of the piecewise linear approximation of the fractiles.

Step 1. Using Theorem 3, we compute γ_i for each $i \in \{1, 2, \dots, N\}$ and find the subset of fractiles that are eligible to be optimal. Let n ($\leq N$) be the number of fractiles eligible for optimality based on this computation.

Step 2. We partition the interval $[a_0, a_m]$ into segments over which a particular fractile is optimal and the model is linear. (Note that there are at most $n + m - 1$ such segments.) We thereby identify the eligible fractiles for each piece $[a_{j-1}, a_j]$ of the feasible price range $[a_0, a_m]$.

Step 3. Using Theorem 1, we identify the optimal unconstrained price for each eligible fractile for each piece.

Step 4. Using Theorem 4, we eliminate dominated fractiles within each piece.

Step 5. Using Theorem 1 to find the optimal return (and associated optimal price and stock level) within the segment for each fractile still eligible to be optimal for each piece, we find the optimal solution by enumeration.

Note that in Step 5, the segment over which the price optimization takes place for a given fractile and piece can be a subset of the interval $[a_{j-1}, a_j]$ over which the linear representation is valid. This explains our formulation of Theorem 1 to allow for this possibility.

5.2. Application to Example 1

We now illustrate our approach using Example 1, with unit cost $c = 3$, salvage value $v = 0.5$, and no penalty cost. In this example, the demand distribution is normal for every price r , and the expected value and standard deviation are calculated using conjoint analysis and empirical data for demand for holiday products. Further details on this example can be found in the appendices and in Raz (2003). Our computations are all done using an approximation of the problem using 20 representative fractiles and five pieces for the piecewise linear approximations of

Figure 6 The Objective Function for Example 3

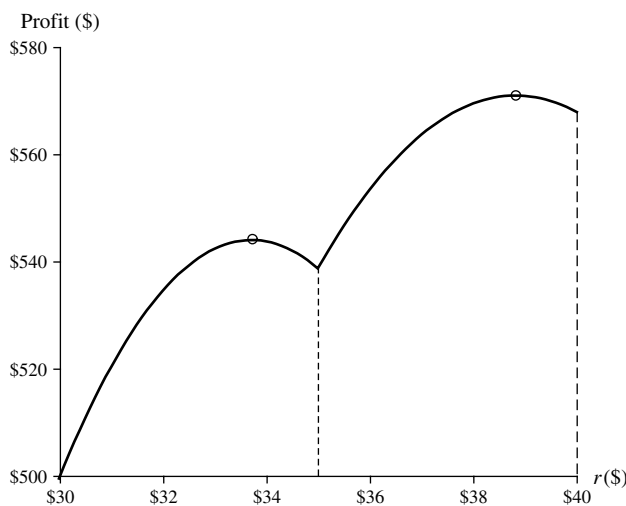
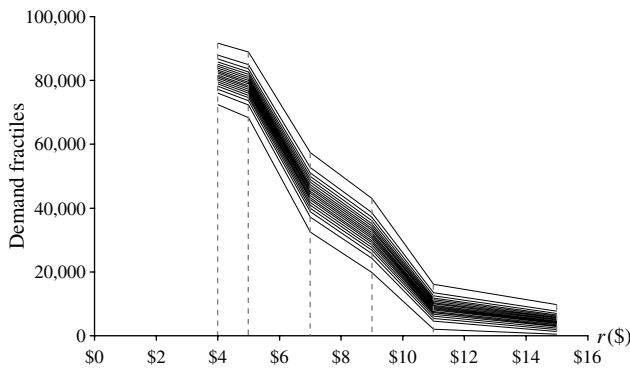


Figure 7 Selected Piecewise Linear Fractiles for Example 1



the fractiles. Figure 7 presents the 20 representative fractiles.

Each of these five pieces has a different approximating linear model and each of these is further subdivided using Theorem 3, which results in 12 different segments to consider, as presented in Figure 8.

The left panel of Figure 8 shows the optimal stocking quantity as a function of the price, as specified by the 12 segments. The right panel plots the objective function over the interval [5, 9.5]. It shows the existence of two local optima. In this case, the local optima arise within pieces with different approximating linear models. The first one is on the 12th fractile using the linear model for the second piece and the second, the global optimal solution, is on the 14th fractile using the linear model for the third piece.

6. Comparative Statics for the Problem

In this section, we examine how the solution changes as a function of the problem parameters. We start by examining the linear fractile problem.

6.1. The Linear Fractile Problem

In this subsection, we consider how changes in the environment influence the firm’s optimal behavior for

the linear fractile problem when restricting the price to be within its eligible segment. Recall that each linear model (corresponding to one piece of the piecewise linear approximation) has the same representation. In particular, the feasible interval $[a, b]$ of prices for a given linear fractile problem in Theorem 1 has the form $[a_{j-1}, a_j]$ for some piece j .

THEOREM 5. *Holding all else constant, the optimal price for the i th fractile problem on piece j , when restricting the price to be within $[\gamma_{i-1}, \gamma_i]$ as well as within $[a_{j-1}, a_j]$:*

- (a) *increases in the unit cost, c , of the product, and*
- (b) *can either increase or decrease as the outcomes become stochastically worse either through lowering one or more fractile functions or changing the probabilities.*

Note that because $y = D_i(r)$, the way in which the quantity (stocked) changes can be determined easily. Part (a) makes intuitive sense: when the unit cost increases, the margin decreases for each price, leading to an increase in the optimal unconstrained price.

The proof of part (b) entails providing examples in which, as the customer environment becomes stochastically worse, the optimal price for a given fractile can either increase or decrease.

6.2. The General Problem

We now examine the effect of changes in the problem environment on the overall optimal behavior of the firm, not just for the linear fractile problem. Specifically, we discuss the effect of changes in the cost on the optimal price and quantity overall. We first show that the optimal quantity is decreasing in the variable cost.

THEOREM 6. *Holding all else constant, the optimal quantity for the problem as a whole, y^* , decreases as the variable cost increases.*

Although the optimal (overall) quantity is monotone decreasing in the unit cost (as can be seen in Figure 9), Figure 2, which is also based on Example 2, indicates that the optimal (overall) price may

Figure 8 The Eligible Optimal Fractiles Based on Theorem 3

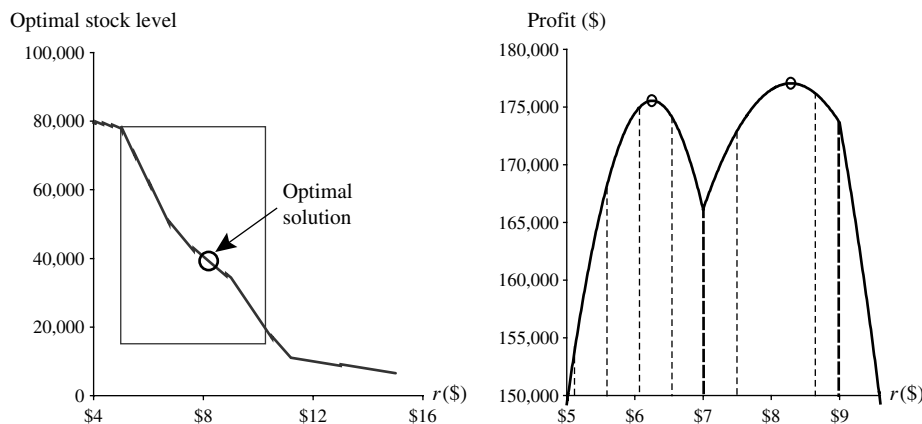
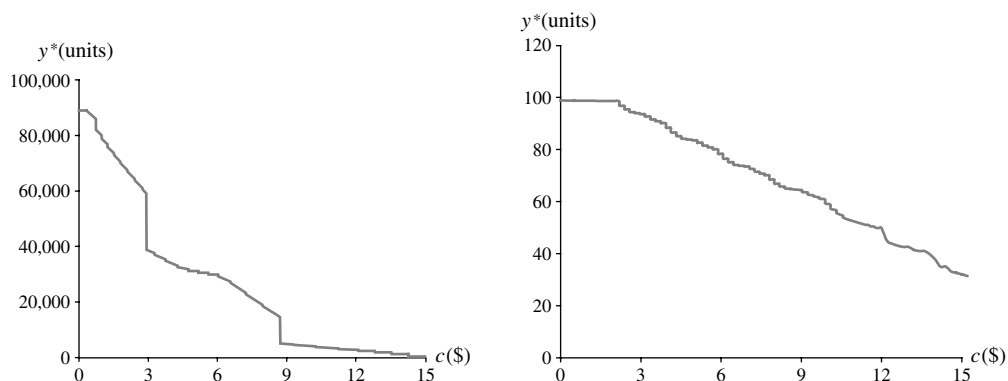


Figure 9 The Optimal Quantity as a Function of the Unit Cost, c



be nonmonotone in the unit cost. This result is not surprising in light of Zabel’s (1970) result that the optimal price may be nonmonotone in the unit holding cost. Before exploring deeper explanations of this behavior, we first examine whether it is simply caused by our discrete approximation. Figure 10 displays the results of analyzing Example 2 with two different discretizations, using 100 and 400 fractiles, respectively. We argue that even finer approximations will further smooth the results, but not change the fundamental phenomenon.

We are grateful to an anonymous referee for the following insight: A higher r corresponds to (a) lower average demand, and (b) higher underage costs. The first suggests a lower stock level and the second suggests the opposite. Indeed, Figure 5 already revealed that the optimal stock level can be a nonmonotone function of the price. Thus, consistent with Zabel’s (1970) result for multiplicative models, the optimal price may also be a nonmonotone function of the stock level.

We now explore the behavior in more detail. By Theorem 3, the boundaries of the segment where a given fractile is optimal may also depend on c . In particular, while the unconstrained optimal price, r_i , increases in c , so do γ_{i-1} and γ_i . It is easy to show that in the linear fractile model, the rate of increase of the γ_i s will be higher than that of r_i (by (17), (22), and (24)). We saw in Figure 5, which corresponds to $c = 7.8$ in Example 2, that the optimal solutions for fractiles 71 and 70 are interior solutions and the optimal solution overall is r_{71} . As c increases, γ_{70} will eventually catch up to r_{71} , and, at that point, by Theorem 4, fractile 70 will dominate fractile 71, and the optimal solution will equal r_{70} , which is strictly smaller than γ_{70} . (The decrease in optimal price to r_{70} from r_{71} will occur before γ_{70} catches up to r_{71} .)

Table 1, which summarizes the results for Example 2 as c is varied in the range of 7.5 to 8.0, using 100 fractiles, indicates that the change to fractile 70 takes place somewhere between 7.8 and 7.9. With a very fine discretization, the optimal fractile would decrease

Figure 10 The Price as a Function of the Cost for Different Discretizations

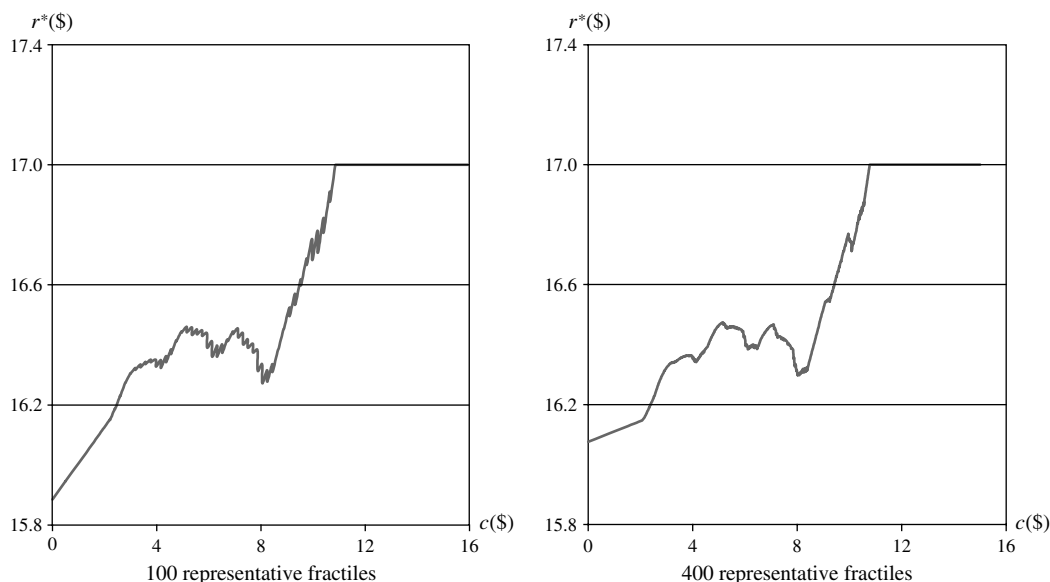


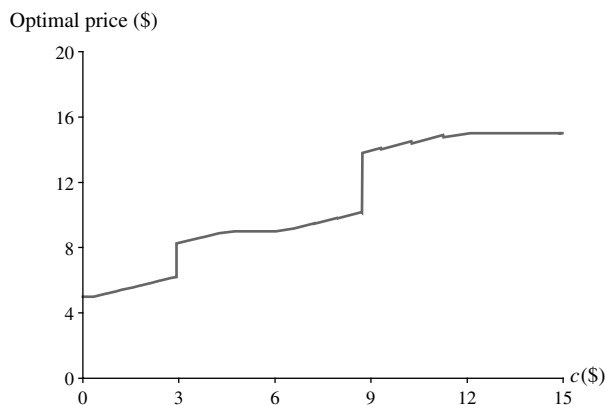
Table 1

c (\$)	$\gamma_i \in [15, 17]$	r^* (\$)	y^*	i^*	Π^* (\$)
7.5	$\gamma_{70} = 15.33, \gamma_{71} = 15.97, \gamma_{72} = 16.64$	16.396	70.750	72	256.05
7.6	$\gamma_{69} = 15.06, \gamma_{70} = 15.67, \gamma_{71} = 16.31$	16.403	70.746	72	248.98
7.7	$\gamma_{69} = 15.39, \gamma_{70} = 16.00, \gamma_{71} = 16.66$	16.378	70.018	71	241.96
7.8	$\gamma_{68} = 15.13, \gamma_{69} = 15.71, \gamma_{70} = 16.33$	16.383	70.016	71	234.96
7.9	$\gamma_{68} = 15.44, \gamma_{69} = 16.03, \gamma_{70} = 16.67$	16.326	68.407	70	228.11
8.0	$\gamma_{67} = 15.18, \gamma_{68} = 15.75, \gamma_{69} = 16.35$	16.274	66.878	69	221.29

quickly through several levels, leading to a reduction in the optimal price.

Figure 11 shows the optimal price as a function of the cost for Example 1, which as we saw in §5, has 12 segments eligible for optimality. (When $c = 3$, there are two local optima in the interval $[4, 15]$ and these are the two shown in Figure 8 in the interval $[5, 9.5]$.) We focus here on the changes when we move from one piece to the next. Note that there are two major jumps, one when the cost hits \$2.92 and a second when it hits \$8.72. When the cost is less than \$2.92, the optimal price is in the interval $[5, 7]$ and the price increases although the optimal fractile decreases, going from the 20th fractile down to the 12th. When the cost reaches \$2.92, the optimal price jumps from \$6.21 to \$8.25 because the optimal price moves from the interval $[5, 7]$ to the interval $[7, 9]$, which, as can be seen in Figure 7, is a move from a high slope interval to a low slope one. That is, we move from an interval where the fractiles are sensitive to the price to one where they are not very sensitive to it. The optimal fractile jumps from the 12th fractile to the 14th. The intuition is that when faced with low sensitivity to the price, the firm can both increase the price and increase its fractile and still be more profitable. Note that although the optimal fractile increases, the quantity decreases, as confirmed in Figure 9 and assured by Theorem 6. When the cost reaches \$8.72, we have a similar phenomenon, jumping from the interval $[9, 11]$ to $[11, 15]$ and from

Figure 11 The Optimal Price as a Function of the Unit Cost



the 3rd fractile to the 8th. The optimal price jumps from \$10.15 to \$13.80. When $c \in [4.52, 6.23]$, we have another interesting phenomena in which the optimal price remains constant, equal to \$9.00. As long as c is lower than \$6.02, we are in the interval $[7, 9]$ and the optimal fractile decreases from the 11th fractile to the 8th, however, the optimal price is achieved on the boundary point, $r = \$9.00$. When $c = 6.02$, we move to the interval $[9, 11]$ and the 7th fractile. However, because we move from a low slope interval to a high slope one, the optimal price remains \$9.00, which is now the lower boundary point. Only when c reaches 6.23 does the optimal price start increasing again.

7. Comparison of the General Model with the Additive and Multiplicative Models

In this section, we compare the results of our model with those of using the additive or the multiplicative demand models for the parameters of Example 1, which as described above, are based on empirical data for a holiday product. Recall from §2 that in the additive case, $D(r) = g(r) + U$, while in the multiplicative case, $D(r) = g(r)W$, where $g(r)$ is a deterministic function that is weakly decreasing in r , U is a random variable with mean zero, and W is a random variable with mean one. For both models, we use a uniform distribution with limits that are based on the variance of the real demand and use the empirical data to derive the function $g(r)$. Figure 12 describes the function $g(r)$ that is used for both the additive and multiplicative approximations.

Figure 13 plots the standard deviation as a function of price for each model, the original model, the additive (approximate) model, and the multiplicative (approximate) model. (In the additive model, the standard deviation is 4,350 for every price, while in the multiplicative model, the coefficient of variation is 0.104 for every price.)

Table 2 gives the optimal price and quantity for several different unit costs for each of three models (the original, the additive, and the multiplicative).

Figure 12 $g(r)$ as a Function of the Price, r

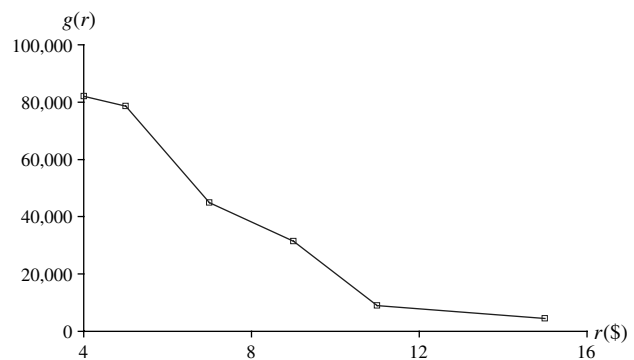
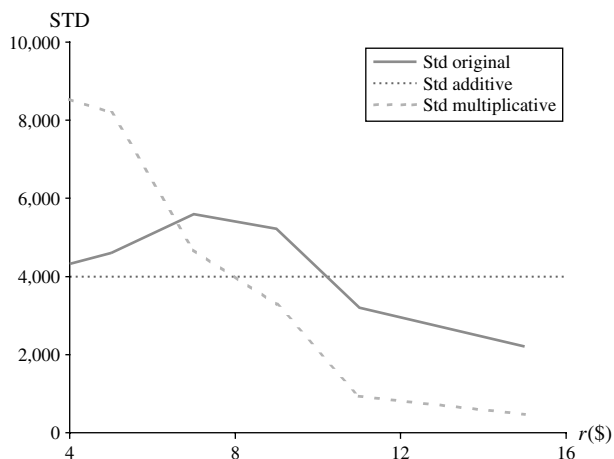


Figure 13 Standard Deviation as a Function of Price for the Three Models



The last two columns give how much lower the profits are if the additive and multiplicative solutions, respectively, are implemented in the original environment. Petruzzi and Dada (1999) argue that the optimal solution will tend to favor prices that have lower demand variability. Furthermore, if an approximate model uses a poor estimate of the demand variability, then its stock level will be misaligned with the price it sets. That logic helps to understand the results: For low values of the unit cost, the additive model does well because it approximates the correct model reasonably well around the optimal solution for that case. However, for unit costs of \$9 or more, the additive model underestimates the reduction in (demand) variability that comes from an increase in price, sets too low a price, uses too large an estimate of the demand variability, and performs poorly. On the other hand, the multiplicative case overestimates the decrease in variability that comes from increasing the price and therefore sets too high a price. It also overestimates the variability when the cost is low ($c = \$2$) and underestimates it when the cost is high ($c \geq \$10$), leading to poor quantity decisions and inferior performance.

Table 2 Comparison Table for Example 1

c (\$)	r^* (\$)	y^*	r_A (\$)	y_A	r_M (\$)	y_M	Π^* general	$\Delta\Pi_A^*$ (%)	$\Delta\Pi_M^*$ (%)
2	6.60	57,586	5.82	67,994	7.92	42,905	238,796	0.03	8.69
3	7.28	48,785	8.28	38,868	8.44	37,582	177,051	0.02	0.10
4	7.97	40,825	8.74	34,298	8.95	32,814	140,744	0.02	0.17
5	8.67	33,649	9.00	30,874	9.00	31,127	108,530	0.04	0.00
6	9.00	29,889	9.00	29,367	9.00	29,793	78,149	0.02	0.00
7	9.37	24,551	9.22	25,448	9.45	24,259	50,746	0.50	0.38
8	9.82	18,316	9.69	19,214	9.93	18,751	29,240	1.15	1.82
9	13.94	4,911	10.15	13,072	14.17	5,239	15,927	16.55	2.47
10	14.37	4,147	10.62	7,004	14.65	4,637	11,441	69.18	6.75
11	14.77	3,405	12.66	1,975	15.00	4,184	7,717	62.47	16.32

8. Conclusion

We have achieved two goals in this paper. The first is to provide a tractable approach to solving quite general forms of the joint price/quantity newsvendor problem, which covers not only problems in which the fractiles are parallel or get closer as the price increases, but ones such as those depicted in Examples 1 and 2. For instance, Example 1 is based on the use of conjoint analysis and empirical data to estimate the demand for fashion holiday products. The result is not an additive model, a multiplicative model, or a mixture of the two. Rather, the demand variability is not monotone in the price, which happens because the price varies from attracting almost all of the market to almost none, with the highest variability arising for prices in the middle. In general, our approach can be applied when a detailed (theoretical or empirical) marketing model of consumer choice is used, without the requirement that the resulting model be a mixture of additive and multiplicative uncertainty. It may also prove to be a useful tool when that requirement is met.

Our second achieved goal is to provide insights into how the solution of our problem depends on the underlying environment. We use a standard approach to approximating a given distribution with a finite number of representative fractiles. We further assume that these fractile functions are piecewise linear functions of the price. The optimal stock level lies on one of the fractile functions. The solution is found by first finding the best price and quantity for the finite number of fractiles that are eligible for optimality, and then choosing the optimal solution by enumeration. The linear fractile problem is well behaved: the objective function is concave in the price, and the optimal price increases and quantity decreases when the unit cost goes up. Our only guarantee that the general problem is well behaved is that the optimal quantity decreases as the unit cost goes up. In particular, we have explained why the optimal price may be non-monotone in the unit cost, which, although implicitly suggested by Zabel (1970), is not intuitively obvious. Furthermore, even the linear fractile problem's solution can be nonmonotone in changes made in the customer environment.

There are a number of possible applications of this research. One of our main motivations for analyzing this model was to build a framework that would enable us to examine supply chain contracting issues for the case of endogenous retail pricing. Raz (2003) reveals some interesting phenomena that can arise when neither the retailer nor the manufacturer have sufficient power to capture the entire chain profits. For instance, when a wholesale price only contract is used, the manufacturer's objective function can have multiple local maxima.

The main limitations of our model are that we approximate a continuous demand model using a finite number of representative fractiles and that the fractile functions are piecewise linear. While this means that our model is not completely general, we argue that by increasing the number of fractiles and/or the number of pieces in the piecewise approximation, we can get an arbitrarily good approximation of almost any continuous distribution. Furthermore, we do not make further assumptions on the underlying form of the demand distribution, which, as mentioned, offers the opportunity to use more general demand models that might be more applicable in practice.

Our approach can be used in cases in which demand is not even first-order stochastically decreasing in price. The main difference is that some of the slope parameters of the fractile functions would not be positive. Only our results that depend on these slope parameters being positive would not hold. The objective function of the linear fractile problem would no longer be guaranteed to be concave. However, it should still be straightforward to find the optimal solution.

We are grateful to an anonymous referee for suggesting another way to approach the solution to our problem: Instead of discretizing the uncertainty and treating price as a continuous variable, we could discretize the price and treat the resulting uncertainty (in demand) as a continuous random variable. This approach has some nice features. First, for each

price, the newsvendor model gives us the optimal fractile immediately. Second, there is no requirement to approximate any fractile functions with piecewise linear functions. Third, there might be a relatively small number of prices that can legitimately be selected, and, thus, enumerating over them would not be computationally burdensome. Further pursuit of this approach, including comparison of the insights obtainable from both approaches, is worthy but beyond the scope of this paper.

Finally, our model assumes that there is no active competition: Any potential competitors are assumed to have acted first and are unresponsive to any price and quantity decisions made in this model. Allowing for active competition would be a useful extension. Raz (2003) discusses how the use of conjoint analysis in estimating the demand model of Example 1 provides the groundwork for finding any Nash equilibria in a price competition among competing substitute products offered by other firms.

An online supplement to this paper is available on the *Management Science* website (<http://mansci.pubs.informs.org/ecompanion.html>).

Acknowledgments

The authors thank Candi Yano (the department editor) the anonymous associate editor, and three referees for their very helpful comments and suggestions. They are particularly grateful to Nick Petrucci who encouraged the authors to explore the connections with Zabel's (1970) results, which led to major improvements in the paper.

Appendix

Demand Distribution for Example 1 with 20 Representative Fractiles and Five Intervals

Data: $c = 3$, $v = 0.5$, $p = 0$, $a_0 = 4$, $a_1 = 5$, $a_2 = 7$, $a_3 = 9$, $a_4 = 11$, $a_5 = 15$, and $p_i = 0.05$ for $i = 1, \dots, 20$.

Curve i	A_{i1}	A_{i2}	A_{i3}	A_{i4}	A_{i5}	k_{i1}	k_{i2}	k_{i3}	k_{i4}	k_{i5}
1	72,386	68,368	32,460	19,807	2,001	4,017	17,954	6,327	8,903	390
2	76,067	72,296	37,229	24,259	4,553	3,770	17,534	6,485	9,853	818
3	77,305	73,618	38,833	25,756	5,475	3,687	17,392	6,538	10,140	881
4	78,200	74,573	39,992	26,839	6,142	3,627	17,290	6,577	10,348	927
5	78,937	75,359	40,947	27,729	6,690	3,578	17,206	6,609	10,519	965
6	79,582	76,048	41,783	28,510	7,171	3,535	17,132	6,636	10,669	998
7	80,171	76,676	42,545	29,222	7,610	3,495	17,065	6,662	10,806	1,029
8	80,723	77,265	43,260	29,889	8,021	3,458	17,002	6,686	10,934	1,057
9	81,251	77,829	43,945	30,528	8,414	3,423	16,942	6,708	11,057	1,084
10	81,767	78,379	44,613	31,151	8,798	3,388	16,883	6,731	11,177	1,110
11	82,278	78,925	45,275	31,770	9,179	3,354	16,825	6,753	11,295	1,137
12	82,794	79,475	45,943	32,393	9,563	3,319	16,766	6,775	11,415	1,163
13	83,322	80,039	46,628	33,032	9,957	3,284	16,706	6,798	11,538	1,190
14	83,874	80,627	47,342	33,700	10,368	3,247	16,643	6,821	11,666	1,219
15	84,462	81,255	48,105	34,411	10,806	3,207	16,575	6,847	11,803	1,249
16	85,108	81,944	48,941	35,192	11,287	3,164	16,502	6,874	11,952	1,282
17	85,845	82,730	49,895	36,083	11,836	3,114	16,418	6,906	12,124	1,320
18	86,740	83,686	51,055	37,165	12,503	3,054	16,315	6,945	12,331	1,366
19	87,978	85,007	52,659	38,663	13,425	2,971	16,174	6,998	12,619	1,429
20	91,659	88,935	57,427	43,114	16,167	2,724	15,754	7,157	13,474	1,618

Demand Distribution for Example 2 with 100 Representative Fractiles

Data: $c = 7.8$, $v = 2$, $p = 5$, $a_0 = 15$, $a_1 = 17$, and $p_i = 0.01$ for $i = 1, \dots, 100$.

Curve i	A_i	k_i	Curve i	A_i	k_i	Curve i	A_i	k_i
1	2.5	1.04	35	35.5	1.45	69	69.7	0.98
2	5.5	2.24	36	36.5	1.70	70	70.5	0.35
3	7.8	3.15	37	37.5	1.88	71	71.5	0.54
4	8.8	3.35	38	38.5	1.75	72	72.5	0.74
5	9.3	3.33	39	39.5	0.88	73	73.5	0.67
6	11.8	4.33	40	40.5	0.35	74	74.5	0.75
7	14.5	5.38	41	45.5	2.54	75	75.5	1.25
8	16.5	5.75	42	51.5	5.24	76	76.5	1.75
9	19.0	5.63	43	54.5	6.53	77	77.5	1.75
10	21.0	5.60	44	57.5	7.73	78	78.5	1.50
11	23.0	6.29	45	60.5	8.95	79	79.5	0.88
12	25.0	6.99	46	63.5	10.20	80	80.5	0.35
13	27.0	7.78	47	65.1	10.68	81	81.5	0.54
14	29.0	8.48	48	65.3	10.15	82	82.5	0.74
15	30.1	8.73	49	65.5	8.88	83	83.5	0.67
16	30.2	8.53	50	65.7	7.95	84	84.5	0.75
17	30.3	8.25	51	65.9	7.74	85	85.5	1.25
18	30.4	7.68	52	66.1	7.54	86	86.5	1.75
19	30.5	6.35	53	66.3	7.08	87	87.5	1.75
20	30.6	5.38	54	66.5	6.75	88	88.5	1.50
21	30.7	5.11	55	66.7	6.85	89	89.5	0.88
22	30.8	4.86	56	66.9	6.95	90	90.5	0.35
23	30.9	4.70	57	67.1	6.55	91	91.5	0.54
24	31.0	4.45	58	67.3	5.90	92	92.5	0.74
25	31.1	4.23	59	67.5	4.88	93	93.5	0.67
26	31.2	4.03	60	67.7	3.95	94	94.5	0.75
27	31.3	3.75	61	67.9	3.74	95	95.5	1.25
28	31.4	3.18	62	68.1	3.54	96	96.5	1.75
29	31.5	1.85	63	68.3	3.08	97	97.5	1.75
30	31.6	0.88	64	68.5	2.75	98	98.5	1.50
31	31.7	0.61	65	68.7	2.85	99	99.5	0.88
32	32.4	0.66	66	68.9	2.95	100	100.0	0.25
33	33.5	1.03	67	69.1	2.55			
34	34.5	1.23	68	69.3	1.90			

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